Criticality in the Burridge-Knopoff model

Ian Clancy* and David Corcoran†
Department of Physics, University of Limerick, Limerick, Ireland
(Received 8 November 2004; published 18 April 2005)

Criticality is a potential origin of the scale invariance observed in the Gutenberg-Richter law for earthquakes. In support of this hypothesis, the Burridge-Knopoff (BK) model of an earthquake fault system is known to exhibit a dynamic phase transition, but the critical nature of the transition is uncertain. Here it is shown that the BK model exhibits a dynamic transition from large-scale stick-slip to small-scale creep motion and through a finite size scaling analysis the critical nature of this transition is established. The order parameter describing the critical transition suggests that the Olami-Feder-Christensen model may be tuned to criticality through its assumptions describing the relaxation of the system.

DOI: 10.1103/PhysRevE.71.046124 PACS number(s): 89.75.Hc, 05.65.+b, 91.30.Bi

I. INTRODUCTION

The Burridge-Knopoff (BK) model was originally proposed [1] as a conceptual representation of an earthquake fault (Fig. 1). The model is a one-dimensional, so-called slider block model, with the single dimension representing large earthquakes that span the depth of the schizosphere and only tend to propagate in one dimension along the fault [2]. The BK model is not the only model of an earthquake fault (see, e.g., Fisher *et al.* [3]), but it or its derivatives have been central to the statistical analysis of model earthquake dynamics in recent years. Numerical studies by Carlson *et al.* [4–9] have demonstrated that the BK model can produce a power-law moment (event size) probability density distribution (PDD) in general agreement with scale invariance in earthquake behavior: the Gutenberg-Richter (GR) law [10]. Klein *et al.* [11,12] have also explored slider block models with long range interactions and while identifying smaller "earthquake" events with fluctuations near a spinodal critical point, larger events are associated with spinodal nucleation. Nakanishi [13] implemented a one-dimensional cellular automaton of the BK model and observed the GR law. The GR law is also observed for another simplified cellular automaton version of the BK model, the Olami-Feder-Christensen (OFC) model. The OFC model is considered by many to be a self-organized critical system.

Regardless of any possible self-organization, the scale-invariant GR law is generally held to be the result of critical behavior [14]. In this regard, Vieira *et al.* [15] have previously reported for the BK model the existence of a critical transition, from stick-slip motion to continuous sliding. However, the issue of dynamic criticality in the BK model has been contested and Vasconcelos [16] has argued, based on a study of a single block system, that the transition is in fact only first order. In this paper, we demonstrate through finite size scaling analysis that the BK model does indeed exhibit a dynamic critical transition, without delocalized events, and explain the discrepancy with earlier work. Further to this, the general validity of the OFC model is discussed in light of the results found here, particularly the claim that it exhibits self-organized criticality.

*Electronic address: ian.clancy@ul.ie
†Electronic address: david.corcoran@ul.ie

II. THE BURRIDGE-KNOPOFF MODEL

The BK model can in nondimensional form be written in terms of the velocity e and shear stress f. For infinitely slow driving, it is described by

$$\dot{e}_i = f_i - \varphi_i\left(\frac{e_i}{v_f}, f_i\right), \tag{1a}$$

$$\dot{f}_i = k_c(e_{i+1} + e_{i-1} - 2e_i) - k_t e_i, \tag{1b}$$

where the subscripts i refer to the ith grid point or block, k_c and k_t are nondimensional forms of the bulk and compressional spring constants, and φ_i describes the friction on the ith block.

The frictional function used here follows Carlson *et al.* [9] and is given by

$$\varphi_i\left(\frac{e_i}{v_f}, f_i\right) = \begin{cases} \dfrac{1-\sigma}{1 + e_i/(1-\sigma)v_f} & \text{if } e_i > 0, \\ (1-\sigma) & \text{if } e_i = 0 \text{ and } f_i \geqslant 1, \\ f_i & \text{if } e_i = 0 \text{ and } f_i < 1 \end{cases}. \tag{2}$$

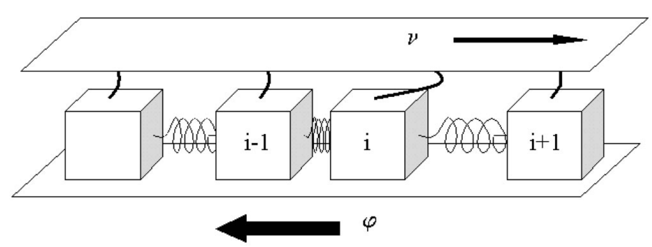

FIG. 1. Schematic of the BK model: a linear array of blocks coupled to nearest neighbors through springs. The blocks rest on a frictional surface with friction φ. Each block is coupled to a rigid plate through a leaf spring, and the plate moves at a constant velocity ν.

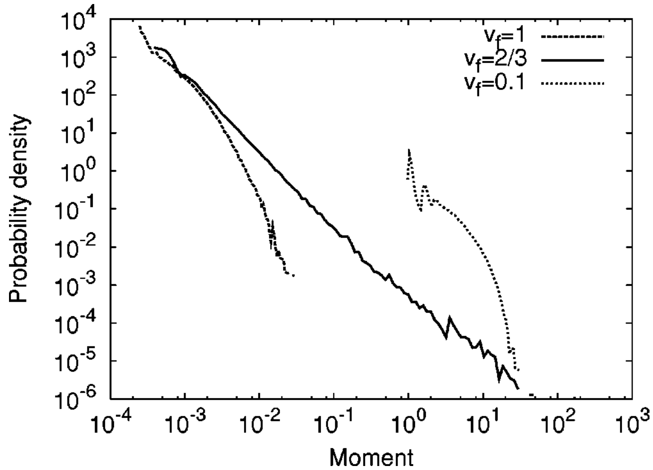

FIG. 2. Moment PDDs below, near, and above the dynamic transition for a 100-block system, with v_f being 0.1, 2/3, and 1.

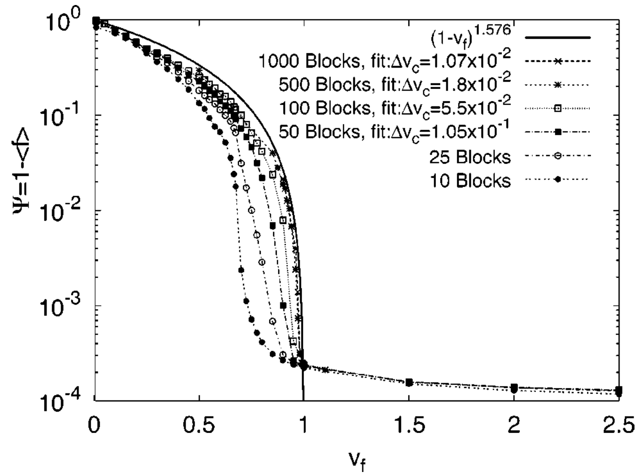

FIG. 3. Plot of Ψ with $\sigma=10^{-4}$ for a number of different size systems. A power law of exponent 1.576 and with a singularity at $v_f=1$ is also shown.

The frictional drop σ allows events to initiate abruptly with an acceleration proportional to σ, and eliminates a strong dependence of initial acceleration on the driving rate [9]. The friction law prohibits slip in the direction opposite to the driving (no back slip).

The system is started from an initially heterogeneous state in which the blocks are attributed random values of small magnitude for each f_i. Driving is manifested by the addition of stress to all blocks of an amount necessary to make the block nearest to the slipping threshold move. The system then relaxes until all e_i are zero and this relaxation constitutes an event. The process of incremental shear stress increase and relaxation is continued for 10^6 iterations for a catalog of 10^6 events.

III. RESULTS

For the experiments $k_c=1$, $k_t=1$, and the frictional drop of Eq. (2), σ, is set to 10^{-4}. Boundary conditions are open and single block events have been omitted in the statistical analysis.

A. A dynamic transition

Figure 2 shows the moment PDDs for three increasing values of the frictional falloff v_f. The form and scale for the distributions changes. It is clear from these changes that a transition in the system dynamics is occurring.

Consider first the largest value of v_f, $v_f=1$. In this case, all events have small moments from about 2×10^{-4} to approximately 2×10^{-2}. The results indicate that the nonlinearity of the friction φ_i is never probed during these events, i.e., block velocities remain very small compared to v_f. The distribution of moments for a 100-block system is fitted well by a stretched exponential (SE) distribution. It is interesting that the PDDs are SE and not simply exponential. SE relaxation has been attributed to a random walk on a fractal [17] and may be an indication of how the attractor for the dynamic phase is being explored.

When the frictional falloff is reduced to $v_f=0.1$ the PDD is over a much larger scale (clearly visible in Fig. 2). If the transition between these two behaviors occurs at approximately $v_f=v_c$ then large-scale stick-slip, or simply stick-slip, is defined as occuring with $v_f<v_c$ and small-scale stick-slip, or creep, is defined as occuring with $v_f>v_c$. With lower v_f an *exponential* distribution is exhibited. Exponential distributions have been observed previously for a 300-block system with $v_f=1/6$ [18], in approximately the same dynamic regime as considered here but without the infinitely slow driving employed in this work.

At or near the value of $v_f=v_c\approx2/3$ a power-law moment probability density distribution, with a small large-scale excess is observed.

B. The transition order parameter

A measure related to the state of the system is the spatially averaged (nondimensional) shear stress $f=(1/N)\Sigma_i f_i$, where f_i is the nondimensional shear stress due to the springs attached to the ith block as defined in Eq. (1). To describe the state of the system it is necessary to obtain the state average $\langle f\rangle$, and this would be expected to attain a statistically steady value, with fluctuations, after a sufficient number of initial states have been visited by the system, the transient period. Indeed, $\langle f\rangle$ does attain a steady state and Fig. 3 shows the behavior of $1-\langle f\rangle$ as v_f is increased from 0.01 to 2.5 for a number of system sizes. The distinction between phases is clear: the creep phase is present while $\langle f\rangle\approx1$, the stick-slip phase when $\langle f\rangle<1$.

In order to resolve the transition region near $\langle f\rangle=1$, the function $\Psi=1-\langle f\rangle$ has been plotted, being the difference between the static frictional threshold and the state averaged shear stress. This also has the advantage that the phase with $v_f>v_c$ is expected to have a value of Ψ near zero, as would be expected of an order parameter. The dynamic transition is clearly visible in the plot of Ψ for different size systems in Fig. 3, and appears to be continuous as would be expected of

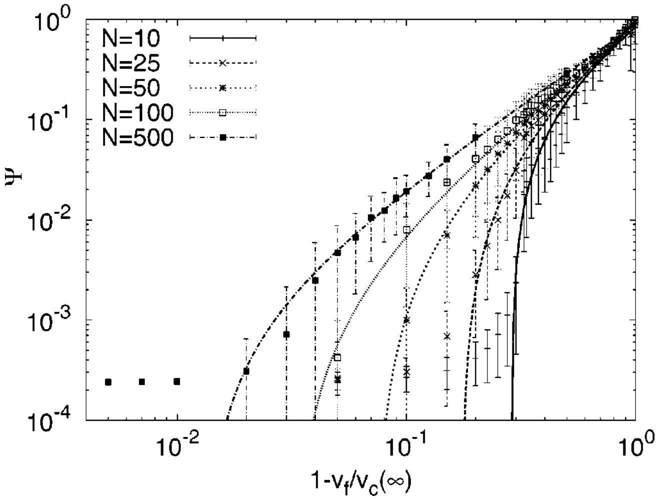

FIG. 4. The unscaled plot of Ψ with $\sigma=10^{-4}$ for different size systems. The fitted curves are power laws shifted by an amount given in Fig. 5 with respect to the true critical point of $v_f=1$.

a higher order transition. Note that Ψ tends to a finite value for $v_f > v_c$. As $\sigma \to 0$, however, it can be shown that $\Psi \to 0$ also. This is because event moment magnitudes would tend to zero and so the system would not move away from the friction stress threshold.

As pointed out earlier it is debatable as to whether the BK model exhibits a first order or continuous dynamic transition. A key method in establishing the criticality of a transition when only a finite size system is available is finite size scaling (FSS).

C. Finite size scaling the order parameter

Figure 3 shows Ψ as a function of v_f for different size systems. The transition point is clearly tending toward $v_f=1$ as the system size N is increased. This transition point of $v_f=1$ was proposed by Vieira *et al.* [15] but later disputed by Vasconcelos [16], the reason being that the relatively fast driving employed in the former work caused this apparent shift. Vasconcelos's argument cannot hold here, however, because in this work an infinitely slow driving mechanism is used and the transition point remains $v_f=1$. A reinforcement of this argument is the finding that the form of friction law used in these works becomes self-similar when $v_f=1$ [19]. This self-similarity means that the microscopic friction imposed on each block is recovered when averaging the stress of friction experienced by the system as a whole, a property to be expected of a scaleless system.

If Ψ is plotted on a log-log scale against $1-v_f$ [or, in general, $1-v_f/v_c(\infty)$ where $v_c(\infty)$ is the critical point of an infinite system], a power-law would be expected. Figure 4 shows such a plot for various system sizes N, but significant deviation from a power law is evident. This deviation is attributed to the shifting of the critical point due to the finite system size. Indeed, power laws of the form $\{1-v_f/[v_c(\infty)-\Delta v_c(N)]\}^{\beta}$ were fitted to the 10-, 25-, 50-, 100-, and 500-block system curves, where $\Delta v_c(N)=v_c(\infty)-v_c(N)$ is the critical point shift. These fits, also shown in Fig. 4, allow an

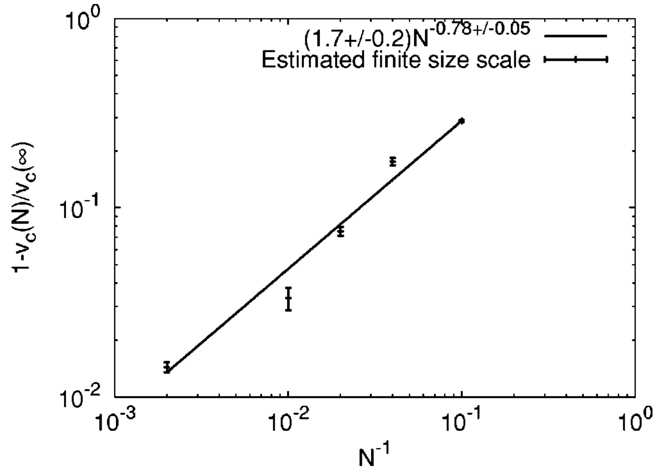

FIG. 5. Critical point shift as a function of inverse system size.

estimate to be made of the critical exponent β and the critical shifting $\Delta v_c(N)$. Using these estimates a FSS analysis can be performed.

To describe the behavior of the system independent of system size one writes [20]

$$\Psi(v,N) = |v|^{\beta}\psi(N/\xi(v)), \quad (3)$$

where $v=[v_f-v_c(N)]/v_c(N)$, $v_c(N)=v_{c(\infty)}-bN^{-1/\nu}$, $v_{c(\infty)}$ refers to the transition point for the infinite block system, and b is a constant. An approximate power-law form of the critical shift is observed (see Fig. 5) yielding $1/\nu=0.78\pm0.05$ and $b=1.7\pm0.2$. The correlation length $\xi(v)$, diverges to infinity as the critical point is approached as a power law $\xi(v) \sim |v|^{-\nu}$, and so multiplying across by $N^{\beta/\nu}$,

$$N^{\beta/\nu}\Psi(v,N) = (N^{1/\nu}|v|)^{\beta}\psi((N^{1/\nu}|v|)^{\nu}). \quad (4)$$

Testing the scaling relation, in Fig. 6 data collapse is observed with $1/\nu=0.78$, $\beta=1.576$, and $v_c(\infty)=1$. The data collapse is within the error associated with each point. A non-

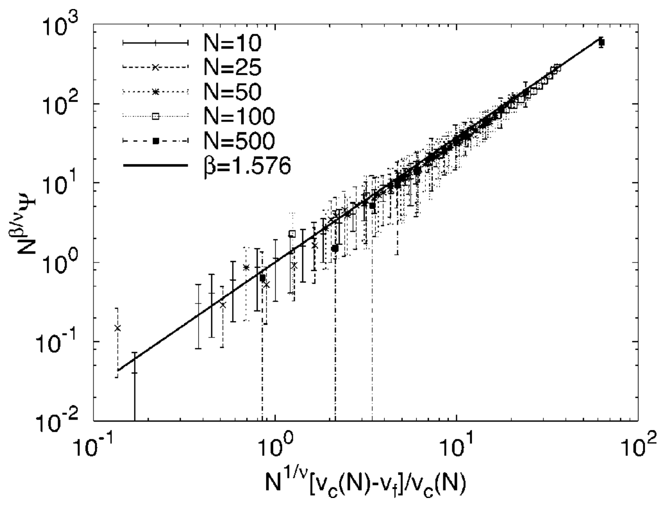

FIG. 6. The finite size scaled plot with $\sigma=10^{-4}$. The functions collapse, within error, with $v_c(\infty)=1$, $1/\nu=0.78$, $\beta=1.576$.

zero value of σ results in the event magnitudes in creep behavior being nonzero, and consequently Ψ being nonzero. As Ψ tends to a finite value, during creep the function diverges from the power law expected of a critical system when Ψ is of the same order of magnitude as σ. This distorts the function near the singularity, and explains the departure from power-law behavior observed in Figs. 3 and 4.

Note that these measured scaling exponents are not trivially related to the system's spatial dimension, as would be expected in the scaling of a first order transition [21]. This nontrivial scaling is consistent with the system exhibiting higher order behavior and so it is expected that even larger systems will obey the same scaling.

It can now be seen that Ψ finite size scales about $v_f=1$, and therefore criticality in the dynamic behavior of the BK model is demonstrated. We now consider claims of the first order, noncritical nature of the BK system, and explore the apparent contradiction.

The results thus far presented have shown that the system apparently displays a continuous transition and has a critical point in the thermodynamic limit and when $\sigma \to 0$. These results contrast with the conclusions reached by Vasconcelos from analysis of a single block system [16] in which it was determined that the transition observed for the single block system is first order and concluded that the same should be true for the multiblock system.

IV. DISCUSSION

A. Comparing the single and multiblock systems

Ψ for a single block system will not collapse onto the FSS curve of Fig. 6. The transition point of the single block system is at $v_f=0.5$; thus $\Delta v_c(1)=0.5$. This point does not fall on the fitted multiblock power-law curve for $\Delta v_c(N)$ shown in Fig. 5. In addition, it is expected that no systems with less than four blocks could fall on the fitted curve $\Delta v_c(N) = (1.7\pm 0.2)N^{-0.78\pm 0.05}$ because $\Delta v_c(N \leqslant 3) > 0.5$.

Data collapse on the multiblock curve would only be expected of critical systems supporting the first order nature of the single block system and opening the possibility of criticality for the multiblock system. It may then be asked why there is such a marked difference between the single and multiblock systems. It is possible, in fact highly likely, that the *interactions* between blocks in the multiblock systems fundamentally affect the dynamics and allow the system to undergo a continuous higher order transition as the results presented here suggest. In fact, it is a vital ingredient of critical systems that interactions between elements of a given model can occur otherwise critical phenomena such as long range correlation have no meaning. As an example, the single Ising spin system is found to have a simple exponential relaxation to an equilibrium state [22], i.e., it does not demonstrate critical behavior. The conclusion from this must be that while the single block system may undergo a first order transition, this is not necessarily the case for a multiblock system, furthermore, the results presented here indicate a higher order transition.

Figure 6 shows that the transition for the infinite system apparently occurs at $v_f=v_c(\infty)=1$, markedly different from the value of $v_f=0.5$ suggested from the study of the single block system. The dependency of the transition on the number of blocks suggests that the degrees of freedom of the system play a vital role. Moreover, it appears that only in the larger systems does finite size scaling apply as critical point shifting obeys a power law only for large systems. This could indicate a transition from first order to higher order behavior dependent on system size.

B. The validity of the OFC model

In interpreting the results of the BK model, the results of similar experimental systems should also be considered. Johansen *et al.* [23] investigated an experimental system consisting of a mass (a work-hardened tool bit) being driven through a spring $\mathcal{S}$ over a frictional surface made from steel. The general result of this study was, depending on the driving rate, normal stress, and spring constant of $\mathcal{S}$, the observation of stick-slip and creep phases in addition to power-law behavior as is observed in this body of work.

In addition, the results produced by Johansen *et al.* suggest that increasing the normal stress (increasing the mass resting on the driven plate) on the contact area (fault) brings about a large-scale-event phase. By rescaling e_i and f_i of Eq. (1) as $z_i=e_i/v_f$ and $F_i=f_i/v_f$, respectively, it can be shown that the tuning parameter v_f can be recast as a parameter controlling the magnitude of the frictional shear stress $\tau = (1/v_f)\varphi_i(z_i)$. But $\tau=\varsigma\mu$, where ς is the normal stress and μ is the coefficient of friction. If φ is identified as the coefficient of friction μ then $\varsigma=1/v_f$ becomes the effective normal stress. A change of the magnitude of τ may be thought of as resulting from a change in the normal stress ς. Thus v_f can be thought of as a parameter controlling ς with $v_f \propto \varsigma^{-1}$. By decreasing v_f, the normal stress increases and the system enters a large event phase in agreement with experiment. It must be noted that there is a discrepancy between the results presented here and those of Johansen *et al.*, who observe the large-scale-event sizes to be Gaussian distributed in contrast to the exponential event moment PDDs observed here.

A small-scale-event phase is also observed by Johansen *et al.* when reducing the normal stress, exhibiting what was described by the authors as an "approximate" exponential distribution of slip event sizes and similar results have been presented here, though here the distribution might be better described as stretched exponential. These small-scale-event PDDs were found to be better fitted as being stretched exponential distributions, i.e., they showed curvature on a log-linear scale. Indeed, it must be pointed out that in the results presented by Johansen *et al.* there also appears to be some curvature in the log-linear plots of the event size PDDs for this phase.

With the existence of stick-slip and creep phases and correspondence in the manner of tuning these phases, the BK model is apparently consistent with laboratory experiments. The OFC model, in contrast, does not show such realistic phases of dynamics. What is the difference between the models? As already mentioned, the OFC model is a cellular automaton version of the BK model; how could it be that the dynamics of the two systems are so different? There are a number of possible answers to this that, taken together, may explain the discrepancy. But first, it may be of benefit to describe the OFC model in some more detail.

1. Comparing the BK and OFC models

The OFC model is a two-dimensional cellular automaton version of the BK model [24]. Each site, labeled as i, j for the ith row and jth column, represents a block resting on a frictional surface connected to its four nearest neighbors through springs, $i\pm 1, j$ and $i, j\pm 1$. As with the BK model, driving is supplied to each block through a leaf spring. Each site has a stress associated with it, $f_{i,j}$. The frictional surface prevents a block from slipping if the stress is less than some threshold stress $f_{\rm th}$. The model's dynamics occur in two stages; buildup, where the stress on all blocks is increased simultaneously; and relaxation where the stress on a block is at or above the stress threshold and is redistributed to the connecting springs and thus nearest neighbors. The buildup stage is actually the same for the OFC and BK models; stress is incremented linearly in time for all elements of the systems until at least one element attains the threshold. The difference between the models is in the relaxation stage.

The BK model solves the equations of motion by applying an appropriate frictional shear stress during the relaxation stage whereas the OFC model sets the shear stress of a relaxing site to zero, $f_{i,j}\rightarrow 0$. This represents the idea that a slipping block will come to rest in the position where the net shear stress on the block is zero, the stress equilibrium. The stress is then reallocated to the four nearest neighbor sites such that $f_{i\pm 1,j}\rightarrow f_{i\pm 1,j}+\alpha f_{i,j}$ and $f_{i,j\pm 1}\rightarrow f_{i,j\pm 1}+\alpha f_{i,j}$. The number α is dependent on the sizes of the spring constants of the compressional spring, k_c, and leaf spring, k_t, and determines the level of conservation of the stress, being fully conservative with $\alpha=0.25$ and completely dissipative with $\alpha=0$. A fully conservative system is not realistic as, from the definition of α, this implies that the leaf spring constant is zero, $k_t=0$, and there can be no driving through the leaf spring. In the case where $k_c=k_t$ and $\alpha=0.2$ the system is nonconservative.

Results of the OFC model show robust power-law behavior in the event size PDD [24]. The power-law exponent of the event size PDD also increases with increasing α, being approximately 2 for $\alpha=0.2$. Hergarten and Neugebauer [25] also observed a power-law decay in excess activity about large events in qualitative agreement with Omori's laws for aftershocks and foreshocks.

2. The role of friction

The primary difference between the BK and OFC models is the friction and how it is modeled. It is the parameter space associated with the friction that has resulted in the wide range of dynamics observed in the BK model. The parameters of the friction are associated with properties relating to the friction such as the normal stress and rate of velocity weakening, properties that can control the phase of the dynamics.

A block experiencing friction corresponding to the large event phase, upon slipping, loses minimal energy to the friction. Its kinetic energy is initially gained from the potential energy stored in connected springs and then lost to be stored again in these springs at which point it comes to rest. In order for the energy in the system to remain finite the energy input must be dissipated to the friction, large events allow the friction to be experienced for long enough to dissipate the required energy. However, for a large event, the block is likely to have overshot its equilibrium stress position, the point at which the net stress on the block due to the springs is zero. Thus these large events cause the system to be, on average, far from threshold and far from the equilibrium stress position. The small event phase can be similarly explained by noting that the dynamic friction remains quite large and so only small events are required to dissipate the energy. The system then creeps by remaining close to threshold throughout the system. The system is again far from the equilibrium stress position.

The OFC model, however, takes a very different approach when dealing with the friction. Friction only really appears in the OFC model as the mechanism to prevent slip: the stress threshold. The dynamics of events are ignored in the OFC model and so the friction does not come into the equations. However, it must be present during events, at least conceptually, as the energy input into the system must be balanced by the energy dissipation on average. The only conceptual energy dissipation in the OFC model is the friction. But the OFC relaxation rule indicates that friction dissipates energy in such a way that a relaxing site always ends its movement in the equilibrium stress position. Of course, from examination of the phases observed in the BK model above, such a relaxation rule disallows these phases from the OFC model. The relaxation rule also indicates that the friction is tuned in that it gives rise to the relaxing site going to the equilibrium stress position, between the behavior expected of stick-slip and creep phases of the BK model that are expected to come to rest far from the equilibrium stress position.

The dynamics of the OFC model may be an accurate representation of the BK model, but only when the relaxation rule is valid, i.e., for parameter values of the friction when a block in the BK model is expected to become stuck at its equilibrium stress position. This is clearly not the case in either of the BK model's stick-slip and creep phases, which overshoot and undershoot the equilibrium stress position, respectively. It may be an interesting modification to the OFC model to change the relaxation rule, possibly allowing these phases to emerge. Could the OFC model, claimed to be a self-organized critical system, in fact be tuned to criticality by an implicit assumption on friction?

In the parameter regime of the BK model between stick-slip and creep phases where power laws are observed, the OFC model relaxation rule may be expected to be valid. In this phase, Olami *et al.* found that it is when $k_c\approx k_t$, a condition used here, that behavior similar to realistic earthquake behavior is observed [24], i.e., the power-law exponent of the event size PDD is approximately 2 (cumulative probability distribution power-law exponent of 1); the same as empirical observations of earthquake behavior. Given that the OFC model is a quasistatic equivalent of the BK model, the results must correspond when the assumptions of the OFC model are valid. This does seem to be the case here, but only under appropriate conditions.

V. CONCLUSIONS

In conclusion the Burridge-Knopoff model has been investigated with $k_c = k_t = 1$ and $\sigma = 10^{-4}$. A dynamic transition has been observed to occur with variation of the tuning parameter v_f. Near the transition, the moment probability density distribution follows a power law and away from the transition exponential and stretched exponential PDDs are observed. The average spatiotemporal shear stress due to neighboring blocks and the driving is seen to attain a steady state for a given v_f of $\langle f \rangle$ that leads to an order parameter in the system. The presence of scaleless fluctuations (power-law moment PDD) through a finite size scaling analysis demonstrate that the system exhibits a critical transition. Furthermore, the Olami-Feder-Christensen model contains an implicit assumption on friction that is here suspected to correspond to the critical point of the Burridge-Knopoff model.

ACKNOWLEDGMENT

We gratefully acknowledge Enterprise Ireland for funding this research.

[1] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. **57**, 341 (1967).
[2] D. L. Turcotte, *Fractals and Chaos in Geology and Geophysics*, 2nd ed. (Cambridge University Press, Cambridge, U.K., 1997).
[3] D. S. Fisher, K. Dahmen, and S. Ramanathan, Phys. Rev. Lett. **78**, 4885 (1997).
[4] J. M. Carlson, Phys. Rev. A **44**, 6226 (1991).
[5] J. M. Carlson, J. Geophys. Res., [Solid Earth] **96**, 4255 (1991).
[6] J. M. Carlson and J. S. Langer, Phys. Rev. A **40**, 6470 (1989).
[7] J. M. Carlson and J. S. Langer, Phys. Rev. Lett. **62**, 2632 (1989).
[8] J. M. Carlson, J. S. Langer, and B. E. Shaw, Rev. Mod. Phys. **66**, 657 (1994).
[9] J. M. Carlson, J. S. Langer, B. E. Shaw, and C. Tang, Phys. Rev. A **44**, 884 (1991).
[10] B. Gutenberg and C. F. Richter, *Seismicity of the Earth and Associated Phenomena* (Princeton University Press, Princeton, NJ, 1954).
[11] W. Klein, J. B. Rundle, and C. D. Ferguson, Phys. Rev. Lett. **78**, 3793 (1997).
[12] C. D. Ferguson, W. Klein, and J. B. Rundle, Phys. Rev. E **60**, 1359 (1999).
[13] H. Nakanishi, Phys. Rev. A **41**, 7086 (1990).
[14] S. Hergarten, *Self-Organized Criticality in Earth Systems* (Springer Press, Berlin, 2002).
[15] M. D. Vieira, G. L. Vasconcelos, and S. R. Nagel, Phys. Rev. E **47**, R2221 (1993).
[16] G. L. Vasconcelos, Phys. Rev. Lett. **76**, 4865 (1996).
[17] P. Jund, R. Jullien, and I. Campbell, Phys. Rev. E **63**, 036131 (2001).
[18] M. D. Vieira, Phys. Rev. E **54**, 5925 (1996).
[19] M. D. Vieira and H. J. Herrmann, Phys. Rev. E **49**, 4534 (1994).
[20] J. Cardy, *Current Physics Sources and Comments*, (Elsevier Science Publishers, Amsterdam, 1988), Vol. 2.
[21] M. E. Fisher and A. Nihat Berker, Phys. Rev. B **26**, 2507 (1982).
[22] H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, London, 1971).
[23] A. Johansen, P. Dimon, C. Ellegaard, J. S. Larsen, and H. H. Rugh, Phys. Rev. E **48**, 4779 (1993).
[24] Z. Olami, H. J. S. Feder, and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992).
[25] S. Hergarten and H. J. Neugebauer, Phys. Rev. Lett. **88**, 238501 (2002).